PHYSICAL REVIEW E **70**, 035602(R) (2004)

Metrics for negative-refractive-index materials

K. J. Webb,[1,2,*] M. Yang,[1] D. W. Ward,[2] and K. A. Nelson[2]
[1]*School of Electrical and Computer Engineering, Purdue University, 465 Northwestern Avenue, West Lafayette, Indiana 47907-2035, USA*
[2]*Department of Chemistry, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139-4307, USA*
(Received 31 March 2004; published 27 September 2004)

The importance of loss in the field solution for left-handed media, and in particular, the impact on what would otherwise be a purely evanescent field, is addressed. Using an equivalent electric current source, field solutions are constructed in semi-infinite and finite thickness left-handed media. In the slab case, field growth and power dissipation metrics with uniform amplitude and uniform power excitation, respectively, provide a means to evaluate the potential of a left-handed material lens. Power dissipation suggests that field growth will be adversely impacted.

DOI: 10.1103/PhysRevE.70.035602 PACS number(s): 42.30.Va, 41.20.Jb, 42.25.Bs, 78.20.Ci

The topic of negative refractive index or left-handed (LH) electromagnetic media, introduced by Vesalago [1], lay dormant until a paper by Pendry suggested that the evanescent fields in such media may grow with distance [2]. It was proposed that the growth of the evanescent fields could compensate for the decay in positive refractive index or right-handed (RH) media, thereby creating the opportunity for building a perfect lens. Interest accelerated after the experimental demonstration of a LH material at microwave frequencies [3]. This material was fabricated using printed circuit board conducting elements that formed electric and magnetic dipoles that were subwavelength in size but which were designed to collectively operate above their resonance. These arrays of metal patterns, used to realize a bulk property, have become known as metamaterials.

In the case of LH media, we associate the phenomenon with one of simultaneous resonances in both the ensemble electric and magnetic field dipole moments, and this implies concomitant absorptive losses [1]. Equivalently, LH media are dispersive, with real and imaginary components of the magnetization (permeability) and polarization (permittivity) being related by the Kramers-Kronig relations [4]. Therefore, any meaningful model of a LH medium must be dispersive and must be lossy. The need to incorporate loss into the field solution in a LH medium slab problem has been identified [5]. To counter the deleterious impact of loss on operation, use of gain media has been proposed [6]. Remaining was the prospect that, subject to interface scattering, fields may be able to grow in a LH medium surrounded by RH media. We provide constraints on this prospect here.

Consider a current sheet in the geometry of Fig. 1 with

$$\mathbf{J}_s = \hat{\mathbf{x}} J_0 e^{ik_{x0}x} \tag{1}$$

on the $z=0$ plane. We assume throughout that the time dependence is $e^{-i\omega t}$. A unique field solution can be obtained using the boundary conditions for the tangential fields at $z=0$, together with radiation conditions for each region, whereby waves having k_z associated with an outward Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}^*$ are selected, where $\mathbf{E}$ is the electric field and $\mathbf{H}$ the magnetic field. The general solution for $\mathbf{H}$ with arbitrary μ_i and ϵ_i is

$$H_{y1} = -J_0 \frac{\dfrac{Z_2}{Z_1}}{\dfrac{Z_2}{Z_1}+1} e^{i(k_{x0}x+k_{z1}z)},$$

$$H_{y2} = J_0 \frac{1}{\dfrac{Z_2}{Z_1}+1} e^{i(k_{x0}x+k_{z2}z)}, \tag{2}$$

where $k_{z_2^1}=\sqrt{k^2-k_{x0}^2}$ and $Z_2^1=k_{z_2^1}/(\omega\epsilon_2^1)$. Given Eq. (2),

$$E_{x_2^1} = \frac{k_{z_2^1}}{\omega\epsilon_2^1} H_{y_2^1}. \tag{3}$$

Consider the lossy case with $\epsilon_1=-\epsilon'+i\epsilon''$ and $\epsilon_2=\epsilon'+i\epsilon''$, where all quantities are positive. The corresponding propagation constants are $k_{z1}=-\beta+i\alpha$ and $k_{z2}=-\beta-i\alpha$, both representing decaying fields away from the source when $\alpha>0$. With $\epsilon''\neq 0$, the boundary condition $\hat{\mathbf{n}}_{21}\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{J}_s$ implies that

$$H_{y2}\big|_{\substack{x=0\\z=0}} = -H_{y1}^*\big|_{\substack{x=0\\z=0}}. \tag{4}$$

The fields in Eq. (2) are thus well behaved as $|z|\rightarrow\infty$, i.e., they decay. As $\epsilon''\rightarrow 0$ and $\beta\rightarrow 0$, $Z_2/Z_1\rightarrow -1$, and $H_{y_2^1}\rightarrow\infty$. In this limit of evanescent fields, $Z_1+Z_2=0$, and the field solution has the character of a resonance. The introduction of loss thus damps this resonance. To satisfy all boundary conditions when regions 1 and 2 are lossless, the evanescent field must grow in one domain, which is nonphysical. The fields in the semi-infinite domains must decay away from the source and they must carry power away from the source.

To address the potential for field growth in a LH material, consider the field solution due to a current source within a

*Electronic address: webb@purdue.edu

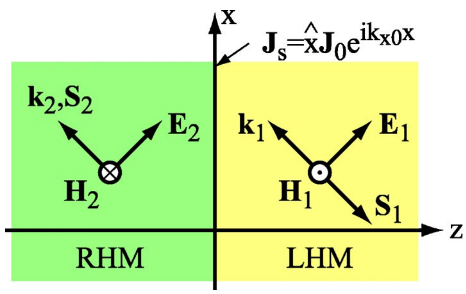

FIG. 1. Current source at the interface between a right-hand (RH) and left-hand (LH) material system.

LH slab of thickness $2d$, as in Fig. 2. The geometry is symmetric, with $0<z<d$ being region 1 and $z>d$ region 3. The solution for $\mathbf{H}_1 = H_{y1}\hat{\mathbf{y}}$ is

$$H_{y1} = Ae^{ik_{x0}x}(e^{ik_{z1}z} + \Gamma_h e^{i2k_{z1}d} e^{-ik_{z1}z}), \tag{5}$$

where Γ_h is the magnetic field reflection coefficient, given by $\Gamma_h = (Z_1 - Z_3)/(Z_1 + Z_3)$, and $A = -J_0[2(1+\Gamma_h e^{i2k_{z1}d})]^{-1}$. Field growth requires $|H_{y1}(d)| > |H_{y1}(0)|$. From Eq. (5), we define the field growth function

$$f = \frac{|(1+\Gamma_h)e^{ik_{z1}d}|^2}{|1+\Gamma_h e^{i2k_{z1}d}|^2}. \tag{6}$$

Field growth therefore occurs for $f>1$, and the field decays for $f<1$.

Consider the case of some degree of loss, resulting in real power flow in region 1 and complex wave impedance $Z_1 = Z_{1r} + iZ_{1i}$. If the loss were small, the solution is perturbed from what would otherwise have purely evanescent fields. With the complex reflection coefficient $\Gamma_h = |\Gamma_h|\exp(i\phi)$, and using Eq. (5) and the corresponding $\mathbf{E}_1$, the Poynting vector expressions evaluated at $z=d$ $[S(d)]$ and at $z=0$ $[S(0)]$ are

$$S(d) = \frac{|A|^2}{2} e^{-2\alpha d}[Z_{1r}(1 - |\Gamma_h|^2) + 2Z_{1i}|\Gamma_h|\sin(\phi)], \tag{7}$$

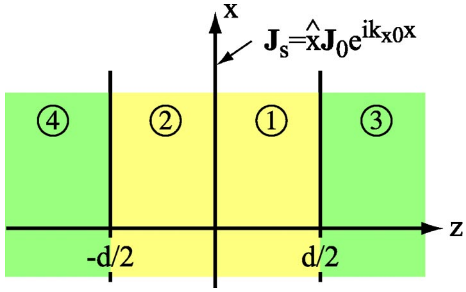

FIG. 2. Current source centrally located in a slab of thickness $2d$.

$$S(0) = \frac{|A|^2}{2}[Z_{1r}(1 - |\Gamma_h|^2 e^{-4\alpha d}) + 2Z_{1i}e^{-2\alpha d}|\Gamma_h|\sin(\phi - 2\beta d)], \tag{8}$$

where $k_{z1} = -\beta + i\alpha$. The conditions $S(d) \leqslant S(0)$, and $S(d) \geqslant 0$ and $S(0) \geqslant 0$ must hold. We consider a normalized Poynting vector of the form

$$S_n(0) = \frac{S(0)}{S(0)|_{k_{x0}=0}}, \tag{9}$$

i.e., $S(0)$ in Eq. (8) is normalized to the power density in the normal propagating wave.

Let the constitutive parameters for regions 1 and 3 be $\epsilon_1 = -\epsilon_1' + i\epsilon_1''$, $\epsilon_3 = \epsilon'$, $\mu_1 = -\mu'$, and $\mu_3 = \mu'$. In this case, $S(d) = 0$, with evanescent fields in region 3. For convenience, we assume that $\mu_1'' = 0$. To facilitate generality, we define the normalized variables

$$a = \frac{k_3}{\alpha_3}, \tag{10}$$

$$\delta = \frac{(\epsilon_1' - \epsilon') - i\epsilon_1''}{\epsilon'} = \delta_r - i\delta_i, \tag{11}$$

$$D = \alpha_3 d, \tag{12}$$

$$p = k_3 d, \tag{13}$$

where $k_3 = \omega\sqrt{\mu'\epsilon'}$ is the wave number in region 3 and, for $k_{x0} > k_3$, $\alpha_3 = |\sqrt{k_3^2 - k_{x0}^2}|$ is the attenuation coefficient for the evanescent field in region 3. Given k_3, p becomes a normalized slab thickness parameter. Also, from Eqs. (10), (12), and (13), $p = aD$. For a given k_3 and ϵ', all field characteristics can be described by the four variables a, δ_r, δ_i, and p or D.

Figure 3 shows f and $S_n(0)$ for low loss [$\delta_i = 10^{-3}$ in Figs. 3(a) and 3(b)] and high loss [$\delta_i = 10^{-1}$ in Fig. 3(c)] as a function of the normalized length parameter p. Figure 3(a) shows the case $a=1$, where $\alpha_3 = k_3$, and Figs. 3(b) and 3(c) give the result for $a = 0.1$. Decreasing a values correspond to evanescent plane waves in region 3 having larger attenuation coefficients, and $a \to \infty$ when a particular plane wave goes through cutoff. Resonant features occur for both $f(p)$ and $S_n(0,p)$. The peak $S_n(0)$ is significantly higher for $a=0.1$, and occurs for smaller values of p. The maximum f occurs for larger p than that for $S_n(0)$. For effective field growth, we require large f and small $S_n(0)$. For a given a there is a range of $f(p)$ where field growth and moderate dissipation can occur. However, comparing the results for $a=1$ and $a=0.1$, it becomes clear that this lower-loss growth regime cannot apply to all evanescent plane waves in region 3.

To understand the impact of the power dissipation in Fig. 3, and to place a measure on $|f|$, we relate the problem of Fig. 2 to the LH lens application. The sheet current source $\mathbf{J}_s(k_x)$ in Fig. 2 is an equivalent source for the fields in the half-space comprised of a lens of thickness d (region 1) and the free space image region (region 3). With planar surfaces, the plane wave spectrum is decoupled, i.e., the scattering problem can be separated into a superposition of solutions

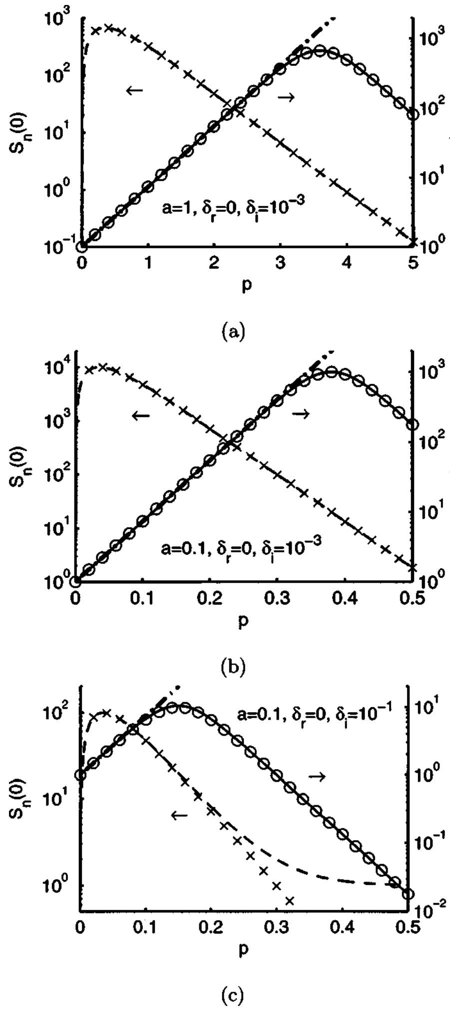

FIG. 3. Normalized Poynting vector $S_n(0)$ [defined in Eq. (9)] and field growth f [from Eq. (6)] as a function of normalized slab thickness p [defined in Eq. (13)] for (a) low loss and a low-order decaying field, (b) low loss and a high-order decaying field, and (c) high loss and a high-order decaying field. Dashed lines: $S_n(0)$. Crosses: perturbational result in Eq. (17). Solid line: f. Dotted-dashed line: required field growth r. Circles: perturbational result in Eq. (20).

for each incident propagating or evanescent plane wave from the object. The finite power from the object can thus be written as the superposition of powers in each plane wave field expansion term. Therefore, the normalized Poynting vector for the true object total field at $z=0$ can be written as

$$S_{on}(0) = \zeta S_n(0), \tag{14}$$

where ζ correctly scales each plane wave solution according to the physical object power density, and $S_n(0)$ is given by (9). The object field is thus

$$H_{oy1} = \zeta^{1/2} H_{y1}, \tag{15}$$

where H_{y1} is given by Eq. (5) and $\zeta^{1/2}=[S_{on}(0)/S_n(0)]^{1/2}$. Consequently, in the solution of the complete scattering problem for each plane wave, those with high $S_n(0)$ in Fig. 3 would have small field amplitudes in the image half-space.

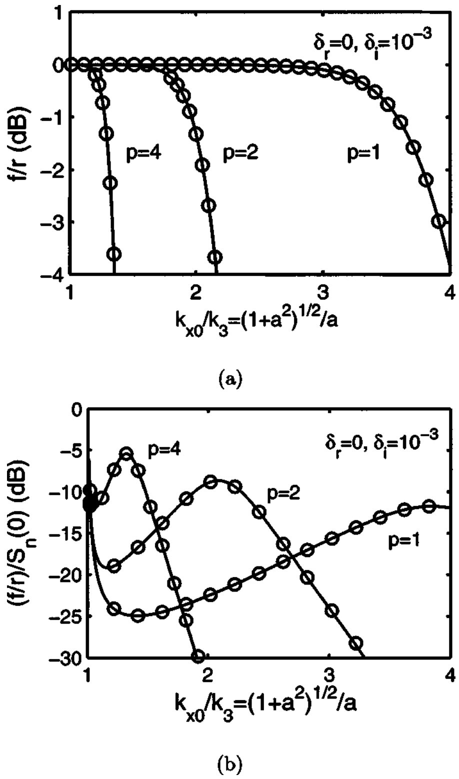

FIG. 4. Spectrum of the decaying field [f from Eq. (6) and r from Eq. (16)] at $z=d$ under two different assumptions with various p, with $\delta_r=0$ and $\delta_i=10^{-3}$, as defined in Eqs. (10)–(13). (a) A uniform current spectrum is assumed at $z=0$. (b) A uniform normalized power spectrum is assumed at $z=0$. Solid lines: numerical simulation. Circles: perturbational result from Eqs. (17) and (20).

Therefore, increasing $S_n(0)$ for fields that do not propagate in region 3 places more of the spectrum below a given detector noise floor.

For a perfect lens, the field growth in the LH lens must compensate for the decay of the evanescent field in the object and image regions. For a lens of thickness d, the sum of the distances from the lens to the object and image planes is also d, and the required field growth can be described by

$$r = e^{2\alpha_3 d}, \tag{16}$$

so that $r=f$ is necessary for each evanescent plane wave to achieve perfect lensing. Figure 3 shows $r(p)$ for each case. In Fig. 3(a), for low loss and a low-order field closer to cutoff, there is a small p window where $f\approx r$ and $S_n(0)$ is not extremely large. However, perfect lensing is precluded under all circumstances when there is any degree of loss.

Figure 4 further explores the impact of loss on the normalized f/r spectrum. Figure 4(a) shows f/r as a function of k_{x0}/k_3, under the assumption of a uniform current spectrum at $z=0$. Increasing p reduces the plane wave spectrum over which $f/r\sim 0$ dB (we could, for example, define a 3 dB bandwidth). To illustrate the impact of the loss further, Fig. 4(b) shows the field spectrum normalized to $S_n(0)$ for each k_{x0} component, i.e., assuming a uniform power spectrum and using the power dissipated in the decaying wave normalized to the power in the normally incident field at $z=0$. Therefore, in this case, $\zeta=S_n^{-1}(0)$. This power normalization more clearly demonstrates the adverse impact of perturbational loss on the potential field growth.

Equation (9), under the assumption that $\delta_r=0$, $a^2\delta_i \ll 1$ (perturbational loss and the plane wave not near cutoff), and $Da^2\delta_i \ll 1$ (sufficiently small thickness), can be approximated as

$$S_n(0) = \frac{8e^{-2D}(1-e^{-2D})}{a(2+a^2)\delta_i}. \tag{17}$$

Using Eq. (17), the maximum of $S_n(0)$ occurs when

$$D_{S_n(0)_{\max}} = \frac{1}{2}\ln 2 \tag{18}$$

or $p=(a/2)\ln 2$, and the value at the maximum is

$$S_n(0)_{\max} = \frac{2}{a(2+a^2)\delta_i}. \tag{19}$$

Use of Eq. (17) predicts the location and value of $S_n(0)_{\max}$ for the parameters used in Fig. 3 nicely. We find it remarkable that the value of $D_{S_n(0)_{\max}}$ is a constant. Under the same assumptions ($\delta_r=0$, $a^2\delta_i \ll 1$ and $Da^2\delta_i \ll 1$), the field growth function in Eq. (6) becomes

$$f = \left[\frac{16}{(2+a^2)^2\delta_i^2}e^{-2D}\right]\left[1+\frac{16}{(2+a^2)^2\delta_i^2}e^{-4D}\right]^{-1}. \tag{20}$$

Using Eq. (20), the maximum of f occurs at

$$D_{f_{\max}} = \frac{1}{2}\ln\frac{4}{(2+a^2)\delta_i}, \tag{21}$$

and the value at the maximum is

$$f_{\max} = \frac{2}{(2+a^2)\delta_i}. \tag{22}$$

From Eqs. (19) and (22), $f_{\max}=aS_n(0)_{\max}$. From $D_{S_n(0)_{\max}}$ and $D_{f_{\max}}$, the $p(=aD)$ window for field growth while keeping low energy dissipation is a function of both the evanescent field attenuation constant (through k_{x0}) and the material loss. The results using the approximations in Eqs. (17) and (20) are shown as circles and crosses in Fig. 3. Figures 4(a) and 4(b) also show the good agreement between the perturbational treatment of Eqs. (17) and (20) (circles) and the numerical simulation (solid lines).

Field growth with propagating fields can occur in either RH, or LH materials when standing waves occur. In the case of decaying fields in LH media, field growth in finite domains is not excluded based on conservation of energy through the Poynting vector. One must therefore conclude that it is plausible that field decay in RH media can be compensated to some degree by LH media with appropriate parameters. The degree to which this can be accomplished is a function of the lens material and geometry, and this can be evaluated using power and field growth metrics. Our understanding of LH material implies that there must be some loss. Unfortunately, any amount of loss in the LH material will preclude the possibility of a perfect lens. It may, however, be possible to build a better lens using LH material. In the case of a curved surface to achieve magnification, by assuming a locally planar geometry, i.e., a large radius of curvature, the conclusions regarding field growth in the planar geometry apply.

ACKNOWLEDGMENTS

This work was supported in part by the Army Research Office under Contract No. DAAD 19-00-1-0387, and the National Science Foundation under Grant No. 0323037-ECS and MRSEC Grant No. DMR-0213282. The authors thank Chiyan Luo for discussions.

[1] V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).
[2] J. B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).
[3] R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001).
[4] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
[5] N. Garcia and M. Nieto-Vesperinas, Phys. Rev. Lett. **88**, 207403 (2002).
[6] S. A. Ramakrishna and J. B. Pendry, Phys. Rev. B **67**, 201101 (2003).